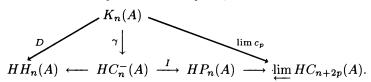# NIL K-THEORY MAPS TO CYCLIC HOMOLOGY

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ABSTRACT. Algebraic K-theory breaks into two pieces: nil K-theory and Karoubi-Villamayor K-theory. Karoubi has constructed Chern classes from the latter groups into cyclic homology. We construct maps from nil K-theory to cyclic homology which are compatible with Karoubi's maps, but with a degree shift. Several recent results show that in characteristic zero our map is often an isomorphism.

Ever since A. Connes discovered cyclic homology in 1982, it has been widely held that cyclic homology is closely related to algebraic K-theory. Supporting this belief, Karoubi defined Chern classes $\lim c_p\colon K_n(A) \to \varprojlim HC_{n+2p}(A)$ in [**K1**], and Goodwillie defined a related map $\gamma\colon K_n(A) \to HC_n^-(A)$ in [**G**]. Other maps between K-theory and variants of cyclic homology have been constructed by Loday (unpublished), Burghelea [**Bur 1**], Ogle [**Ogle 1**], and others. Of course, all of these maps are related to the Dennis trace map to Hochschild homology, $D\colon K_n(A) \to HH_n(A)$, which R. K. Dennis discovered in 1975 [**D**].

In this paper, we prove three results. The first is that Karoubi's map factors through Goodwillie's.

THEOREM A. *For every associative algebra, there is a commutative diagram:*

$$
\begin{array}{ccccc}
 & & K_n(A) & & \\
 & {\scriptstyle D}\swarrow & {\scriptstyle \gamma}\big\downarrow & \searrow {\scriptstyle \lim c_p} & \\
HH_n(A) & \longleftarrow & HC_n^-(A) \xrightarrow{\;I\;} HP_n(A) & \longrightarrow & \varprojlim HC_{n+2p}(A).
\end{array}
$$

We will recall the definitions of HC^-, HP and HC in §1, and prove Theorem A in §2.

In order to explain the rest of our results, we need to decompose algebraic K-groups into two pieces: the Karoubi-Villamayor groups KV_* and the nil K_* groups, which may be thought of as the third terms in a long exact sequence

$$\cdots KV_{n+1}(A) \to \operatorname{nil} K_n(A) \to K_n(A) \to KV_n(A) \cdots .$$

When A is regular, $K_*(A) = KV_*(A)$ and $\operatorname{nil} K_*(A) = 0$, so intuitively $\operatorname{nil} K_*(A)$ measures the contribution to $K_*(A)$ coming from the singularities of A. The example $A = \mathbb{Z}$ shows that the abelian group structure of $KV_*(A)$ can be quite complicated. On the other hand, the nil K-groups are more predictable.

THEOREM B. *If $\mathbb{Q} \subset A$ then* nil $K_n(A)$ *is a $\mathbb{Q}$-module for all n. On the other hand, if $p^r = 0$ in A for some r then* nil $K_n(A)$ *is a p-group for all n.*

For reasons of exposition, we prove this result in §6, because it requires tools unrelated to the main body of this paper.

Our third result is that, if we restrict to $\mathbb{Q}$-algebras, we can decompose Goodwillie's map into two pieces, introducing maps

$$\text{ch}: KV_n(A) \to HP_n(A), \quad \nu: \text{nil } K_n(A) \to HC_{n-1}(A).$$

Before stating this result formally as Theorem C, let us interpret the decomposition informally. We mean that Karoubi's map factors through ch in the sense that

(0.1)

$$
\begin{array}{ccccc}
K_n(A) & \longrightarrow & KV_n(A) & \xrightarrow{\;\text{ch}\;} & HP_n(A) \\
& \searrow & & & \downarrow \\
& \text{lim } c_p & & & \varprojlim HC_{n+2p}(A)
\end{array}
$$

commutes, and ν lifts the Dennis trace map in the sense that

(0.2)

$$
\begin{array}{ccc}
\text{nil } K_n(A) & \longrightarrow & K_n(A) \\
\downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle D} \\
HC_{n-1}(A) & \xrightarrow{\;B\;} & HH_n(A)
\end{array}
$$

commutes. When $n = 1$, the diagram (0.2) is easy to understand.

EXAMPLE 0.3. The image of nil $K_1(A)$ in $K_1(A) = \text{GL}(A)/E(A)$ is

$$\text{Unip}(A)/E(A),$$

where $\text{Unip}(A)$ is the subgroup of $\text{GL}(A)$ generated be all unipotent matrices. The map

$$\nu: \text{nil } K_1(A) \to HC_0(A) = A/[A, A]$$

factors through the trace of the logarithm map

$$\ln(1 + x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 \cdots,$$

in the sense that ν is the bottom row of the diagram

$$
\begin{array}{ccc}
\text{Unip}(A) & \xrightarrow{\;\ln\;} & M(A)/[M(A), M(A)] \\
\downarrow & & \cong \downarrow{\scriptstyle \text{trace}} \\
\text{nil } K_1(A) \longrightarrow \text{Unip}(A)/E(A) & \longrightarrow & A/[A, A]
\end{array}
$$

For example, let $A = \mathbb{Q}[t]/(t^m = 0)$. Then $K_1(A) = \mathbb{Q}^* \oplus \text{nil } K_1(A)$, nil $K_1(A) = (1 + tA)^\times$, and the map B from $HC_0(A) = A$ to $HC_1(A) = \Omega_A$ is the usual derivative map $B(a) = da$. If $x \in tA$ then $D(1 + x) = (1 + x)^{-1}dx = d(\ln(1 + x))$, so by (0.2) we see that $\nu(1+x) = \ln(1+x)$. In this case ν is the familiar isomorphism $\ln: (1+tA)^\times \xrightarrow{\cong} tA$. In fact, by (6.3) this is the universal example of ν when $n = 1$.

The assertions (0.1) and (0.2) follow in general from

THEOREM C. *For every $\mathbb{Q}$-algebra A, there is a commutative diagram with exact rows*

$$
\begin{array}{ccccccc}
\cdots KV_{n+1}(A) & \longrightarrow & \operatorname{nil} K_n(A) & \longrightarrow & K_n(A) & \longrightarrow & KV_n(A)\cdots \\
\text{ch}\downarrow & & \nu\downarrow & & \gamma\downarrow & & \text{ch}\downarrow \\
\cdots HP_{n+1}(A) & \xrightarrow{\ S\ } & HC_{n-1}(A) & \xrightarrow{\ B\ } & HC_n^-(A) & \xrightarrow{\ I\ } & HP_n(A)\cdots \\
\downarrow & & \| & & \downarrow & & \downarrow \\
\cdots HC_{n+1}(A) & \xrightarrow{\ S\ } & HC_{n-1}(A) & \xrightarrow{\ B\ } & HH_n(A) & \xrightarrow{\ I\ } & HC_n(A)\cdots
\end{array}
$$

We will prove Theorem C in §5, after showing via examples in §3 that Theorem C cannot hold integrally. (§4 is devoted to a translation of Theorem A into a topological setting that can be used to prove Theorem C.)

In the remainder of the introduction, I would like to explain why I think the map ν is fundamentally interesting. The first concrete hint that algebraic K-theory and cyclic homology were linked was the Loday-Quillen Theorem [**LQ**, 1.6]. If A is an associative $\mathbb{Q}$-algebra and $\mathfrak{gl}(A)$ denotes the Lie algebra associated to $\mathrm{GL}(A)$, this theorem states that

$$HC_{n-1}^{\mathbb{Q}}(A) \cong \operatorname{Prim} H_n^{\mathrm{Lie}}(\mathfrak{gl}(A); \mathbb{Q})$$

in analogy to the Milnor-Moore theorem, which asserts that

$$K_n(A) \otimes \mathbb{Q} \cong \operatorname{Prim} H_n(\mathrm{GL}(A); \mathbb{Q}).$$

This hint already indicates the shift in indexing between K-theory and cyclic homology that we see in ν. Based on this, R. Staffeldt was able in [**S**] to establish that if $A = R \oplus I$, R the ring of integers in a number field and I nilpotent, then

$$(0.4) \qquad K_n(A, I) \otimes \mathbb{Q} \cong HC_{n-1}^{\mathbb{Q}}(A \otimes \mathbb{Q}, I \otimes \mathbb{Q}).$$

Goodwillie spectacularly generalized this in [**G**], proving that if A is a simplicial ring and I a simplicial ideal with $\pi_0(I)$ nilpotent, then again there is an isomorphism (0.4). The key to this result was the construction of the map γ.

When A is a $\mathbb{Q}$-algebra and I a nilpotent ideal, Goodwillie's theorem tells us more. By [**W2**, p. 301] the groups $KV_n(A, I)$ vanish, so by Theorem B (or [**W3**, 1.4]) the groups $K_n(A, I)$ are $\mathbb{Q}$-vector spaces. In [**G1**], Goodwillie proved that the $HP_n(A, I)$ vanish as well. Thus Goodwillie's result becomes

THEOREM (GOODWILLIE [**G**]). *When I is a nilpotent ideal in a ring A, then $K_n(A, I) \otimes \mathbb{Q} \cong K_n(A \otimes \mathbb{Q}; I \otimes \mathbb{Q})$. When A is a $\mathbb{Q}$-algebra there are isomorphisms*

$$
(0.5) \qquad
\begin{array}{ccc}
\operatorname{nil} K_n(A, I) & \cong & K_n(A, I) \\
\nu\downarrow \cong & & \cong\downarrow\gamma \\
HC_{n-1}(A, I) & \cong & HC_n^-(A, I)
\end{array}
$$

Another result, joint work with C. Ogle [**OW**], concerns the so-called "excision situation": is an ideal of A, and $A \to B$ maps I isomorphically onto an ideal of B. By [**W2**], the double relative groups $KV_n(A, B, I)$ are zero; when A is a $\mathbb{Q}$-algebra, the groups $K_n(A, B, I)$ are $\mathbb{Q}$-vector spaces by Theorem B (or [**W3**, 1.5]). Thus there is a map of $\mathbb{Q}$-vector spaces

$$\nu: K_n(A, B, I) \cong \operatorname{nil} K_n(A, B, I) \to HC_{n-1}(A, B, I).$$

Using the Loday-Quillen Theorem, Ogle and I have proven in [**OW**] that when $\mathbb{Q} \subset A$ and $B = A/J$ then indeed

$$\nu \colon K_n(A, B, I) \cong HC_{n-1}(A, B, I).$$

Third of all, consider the groups $NK_n(A)$, which are the cokernels of $K_n(A) \to K_n(A[t])$. We know that $KV_n(A) = KV_n(A[t])$ and that when $\mathbb{Q} \subset A$ the groups $NK_n(A)$ are $\mathbb{Q}$-vector spaces. When $\mathbb{Q} \subset A$ we also know (e.g. by [**Kass**] or 3.2 below) that $HP_n(A) = HP_n(A[t])$. It follows that we have a diagram of $\mathbb{Q}$-vector spaces, in what should be self-evident notation:

$$\text{nil}\, NK_n(A) \;\cong\; NK_n(A)$$

$$N\nu \downarrow \qquad\qquad N\gamma \downarrow$$

$$NHC_{n-1}(A) \cong NHC_n^-(A)$$

I do not know how often the map $N\nu \cong N\gamma$ is an injection. It clearly is not always an isomorphism. For example, if A is a field of characteristic zero then $NK_n(A) = 0$ for all n, but $NHC_0(A) = A[t]/A$ is not zero. In fact, by [**Kass**, 4.3]

$$NHC_n(A) \cong \Omega_A^n \otimes t\mathbb{Q}[t]$$

can be nonzero for all n, depending on the transcendence degree of A.

I originally constructed the maps ch and ν in March 1985 during the seminar [**IAS**]. I owe C. Kassel a debt of thanks for his encouragement. Subsequently, I discovered that ch was implicit in [**K2**] and that Theorem C could be interpreted as a form of Theorem 7 of [**CK**], where nil K is a case of K^{rel}. Since my viewpoint is much more elementary, and since this work motivated [**OW**], I hope the reader does not mind the overlap. I have tried to backtrack and annotate this paper with literature references in partial atonement.

1. Cyclic modules and their homology. In this section we recall some elementary facts about cyclic modules, establishing notation. It is difficult to give proper credit for these observations; I learned much of this material in the seminar [**IAS**], and from various conversations. I have tried to cite the literature after the fact, and apologize for any omissions. The main references are [**C, K, K1, Bur, Bur 1, Gl** and **G**].

Cyclic modules and their homology are generalizations of simplicial modules and their homotopy. For the sake of uniform notation let us agree that if L is a simplicial k-module, then $L.$ will denote the chain complex with differential $b = \sum (-1)^i d_i \colon L_n \to L_{n-1}$, and $HH_*(L)$ will denote the homology of $L..$. It is well known that $\pi_*(L) = HH_*(L)$ [**May**].

We shall be particularly interested in the simplicial modules $\text{Cyc}(A; M)$ attached to a k-algebra A and an A-bimodule M. Writing $\otimes$ for $\otimes_k$ and $A^{\otimes n}$ for $A \otimes \cdots \otimes A$ (n times), we have $\text{Cyc}_n(A; M) = M \otimes A^{\otimes n}$. We have

$$s_i(m \otimes a_1 \otimes \cdots \otimes a_n) = m \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots,$$

$$d_i(m \otimes a_1 \otimes \cdots \otimes a_n) = \begin{cases} ma_1 \otimes \cdots \otimes a_n & \text{if } i = 0, \\ m \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots & \text{if } 0 < i < n, \\ a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} & \text{if } i = n. \end{cases}$$

The groups $HH_*(\mathrm{Cyc}(A;M))$ are usually denoted $H_*(A;M)$ or $H_*^k(A;M)$. They are the Hochschild homology groups of A with coefficients in M. (See [**CE**, **Mac**].)

DEFINITION 1.1 [**C**, **Bur**, **G1**]. A simplicial object L in some category is called a *cyclic object* if each L_n is given an automorphism t of order $n+1$ satisfying the following combinatorial identities:

$$d_i t = \begin{cases} d_n & \text{if } i = 0, \\ td_{i-1} & \text{if not,} \end{cases} \qquad s_i t = \begin{cases} t^2 s_n & \text{if } i = 0, \\ ts_{i-1} & \text{if not.} \end{cases}$$

(The relations for $d_0 t$ and $s_0 t$ are redundant.)

The homology modules $HC_*(L), HP_*(L)$ and $HC_*^-(L)$ associated to a cyclic module L were introduced in [**C**, **LQ**, **Bur**, **G** and **G1**]. We shall recapitulate their construction shortly. First, however, we shall give some examples and derive some easy facts about them based on maps of cyclic modules.

EXAMPLE 1.2.1 [**C**; **G1**, II.1]. $\mathrm{Cyc}(A;A)$ becomes a cyclic k-module once we set

$$t(a_0 \otimes \cdots \otimes a_n) = a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.$$

Because it is so common, we shall write $HH_*(A)$ for the more awkward $H_*(A;A)$ or $HH_*(\mathrm{Cyc}(A;A))$. Similarly, we shall write $HC_*(A)$ for $HC_*(\mathrm{Cyc}(A;A))$, and so on.

EXAMPLE 1.2.2. Let M be a right kG-module and let $_\varepsilon M$ denote the kG-bimodule whose underlying right module is M, but which is a trivial left kG-module, i.e., $gm = m$ for all g in G. If we define t by the formula

$$t(m \otimes g_1 \otimes \cdots \otimes g_n) = (mg_1 \cdots g_n) \otimes (g_1 \cdots g_n)^{-1} \otimes g_1 \otimes \cdots \otimes g_{n-1}$$

then $\mathrm{Cyc}(kG; {}_\varepsilon M)$ becomes a cyclic k-module. Again for simplicity, we shall write $HH_*(G;M)$ for $HH_*(\mathrm{Cyc}(kG; {}_\varepsilon M))$, $HC_*(G;M)$ for $HC_*(\mathrm{Cyc}(kG; {}_\varepsilon M))$, and so on (cf. [**K**]).

The notation in 1.2.2 is motivated by the classical fact that

(1.2.3) $$HH_*(G;M) \equiv H_*^k(kG; {}_\varepsilon M) = H_*(G;M)$$

(see [**CE**, X.2.1; **Mac**, p. 291]). Here $H_*(G;M)$ denotes the Eilenberg-Mac Lane homology of G with coefficients in the G-module M.

EXAMPLE 1.3.1 (MORITA INVARIANCE). The inclusion of A in the matrix ring $M_r A$ induces a cyclic map from $\mathrm{Cyc}(A;A)$ to $\mathrm{Cyc}(M_r A; M_r A)$. There is a cyclic map going in the other direction:

$$\mathrm{trace}\colon \mathrm{Cyc}_n(M_r A; M_r A) \to \mathrm{Cyc}_n(A; A)$$

$$g^0 \otimes \cdots \otimes g^n \mapsto \sum g_{i_0 i_1}^0 \otimes g_{i_1 i_2}^1 \otimes \cdots \otimes g_{i_n i_0}^n.$$

A priori, this shows that $\mathrm{Cyc}(A;A)$ is a direct summand of $\mathrm{Cyc}(M_r A; M_r A)$ in the category of cyclic k-modules.

In fact, we know that "trace" induces isomorphisms in Hochschild homology [**DI**, 3.7], as well as in HC [**LQ**, 1.7], HP and HC^- [**G**, 1.3.8].

EXAMPLE 1.3.2. It is a classical fact that $H_*(G;k)$ is a direct summand of $HH_*(kG)$. For example, this is implicit in [**CE**, X.2.1] and [**Mac**, p. 291], because k is a summand of the right G-module $_\chi(kG)$.

In fact, $\mathrm{Cyc}(kG; {}_\varepsilon k)$ is naturally a direct summand of $\mathrm{Cyc}(kG; kG)$ in the category of cyclic k-modules.

Consequently, if HX denotes any of the homology theories HH, HC, HP or HC^- then $HX_*(G; k)$ is naturally a direct summand of $HX_*(kG)$. The classical fact cited above follows from this and (1.2.3).

Although this observation is new, the proof is not. It is explicit in [**D**], and dates back to the 1950's. Variations of this observation may be found in [**K** and **Bur**]. One defines maps for each n,

$$\mathrm{Cyc}_n(kG; {}_{\varepsilon}k) \xrightarrow{\iota} \mathrm{Cyc}_n(kG; kG) \xrightarrow{\pi} \mathrm{Cyc}_n(kG; {}_{\varepsilon}k),$$

$$1 \otimes g_1 \otimes \cdots \otimes g_n \mapsto (g_1 \cdots g_n)^{-1} \otimes g_1 \otimes \cdots \otimes g_n,$$

$$g_0 \otimes g_1 \otimes \cdots \otimes g_n \mapsto \begin{cases} 1 \otimes g_1 \otimes \cdots \otimes g_n & \text{if } g_0 = (g_1 \cdots g_n)^{-1}, \\ 0 & \text{if not.} \end{cases}$$

Next, one verifies that ι and π are cyclic maps, and that $\pi\iota$ is the identity. The result follows.

Of the various homology groups associated to a cyclic module L, $HC_*(L)$ is best known. As in [**C**, **G1** or **LQ**], there is a first quadrant double complex L^+_{pq} ($p \geq 0$ and $q \geq 0$) such that $HC_n(L) = H_n(\mathrm{tot}(L^+_{..}))$. The even columns of $L^+_{..}$ are each the chain complex $(L., b)$; the odd columns are each the chain complex $(L., -b')$, where $b' : L_q \to L_{q-1}$ is the map $b' = \sum_{i \neq q}(-1)^i d_i$. The horizontal maps are either $1 - T$ or $N = 1 + T + \cdots + T^q$, $T = (-1)^q t$, depending on the parity of the columns.

The same prescription gives an upper half-plane double complex $L_{..}$ ($q \geq 0$), and a second quadrant double complex $L^-_{..}$ ($p \leq 0$, $q \geq 0$). $HP_*(L)$ is defined to be the homology of $\mathrm{tot}(L_{..})$ and $HC^-_*(L)$ is defined to be the homology of $\mathrm{tot}(L^-_{..})$. Note that the total complex of $L_{..}$ is given by

$$\mathrm{tot}(L_{..})_n = \prod_{p+q=n} L_{pq}.$$

It is easy to compute HP in terms of HC. Since the columns of $L_{..}$ are periodic, we can write $L_{..} = \varprojlim L^+_{..}$; from this it follows that there is an exact sequence

$$(1.4) \qquad 0 \to \varprojlim{}^1 HC_{n+2p}(L) \to HP_n(L) \to \varprojlim HC_{n+2p}(L) \to 0.$$

The HC^- groups bear the same relation to HP and HC that the HH groups bear to HC. That is, we can copy the proof of [**LQ**, 1.6], mutatis mutandis, to prove the following result (see [**C** or **G**, I.3.1]).

PROPOSITION 1.5 (CONNES-GYSIN SEQUENCES). *If L is a cyclic k-module, there is a map of long exact sequences:*

$$\cdots \xrightarrow{S} HC_{n-1}(L) \xrightarrow{B} HC^-_n(L) \xrightarrow{I} HP_n(L) \xrightarrow{S} HC_{n-2}(L) \xrightarrow{B} \cdots$$

$$\Big\| \qquad\qquad \Big\downarrow \qquad\qquad \Big\downarrow \qquad\qquad \Big\|$$

$$\cdots \xrightarrow{S} HC_{n-1}(L) \xrightarrow{B} HH_n(L) \xrightarrow{I} HC_n(L) \xrightarrow{S} HC_{n-2}(L) \xrightarrow{B} \cdots$$

COROLLARY 1.5.1. *If $n \leq 0$ then*

$$HC^-_n(L) = HP_n(L),$$

and $HC^-_1(L) \to HP_1(L)$ is onto.

EXAMPLE 1.5.2. When $L = \text{Cyc}(k; k)$, we see from [**LQ**] that

$$HH_n(k) = \begin{cases} k & \text{if } n = 0, \\ 0 & \text{if not,} \end{cases} \qquad HC_n(k) = \begin{cases} k & \text{if } n \text{ is even, } n \geq 0, \\ 0 & \text{if not.} \end{cases}$$

It follows easily that

$$HP_n(k) = \begin{cases} k & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \qquad HC_n^-(k) = \begin{cases} k & \text{if } n \text{ is even, } n \leq 0, \\ 0 & \text{if not.} \end{cases}$$

The Connes-Gysin sequences break up because $B = 0$, and the vertical maps in 1.5 are isomorphisms for $n \geq 0$.

EXERCISE 1.6. Let $_{\varepsilon}kG$ denote the right kG-module kG, made into a bimodule with G acting trivially on the left. By 1.2.3 we know that

$$HH_n(G; kG) = H_n(G; kG) = \begin{cases} k & \text{if } n = 0, \\ 0 & \text{if not.} \end{cases}$$

Show that the double complex $\text{Cyc}_{..}(kG; _{\varepsilon}kG)$ is a free G-resolution of the chain complex $K. : \cdots \leftarrow k \leftarrow 0 \leftarrow k \leftarrow 0 \leftarrow k \leftarrow \cdots$. (Here at last we allow G to act on the left of $_{\varepsilon}kG$.) Conclude from this that

$$HP_n(G; kG) = \begin{cases} k & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Finally, compare this exercise with the discussion on p. 382 of [**K**].

It is time for our main calculation.

THEOREM 1.7 (KAROUBI [**K**]). *We have*

$$HH_n(G; k) \cong H_n(G; k),$$

$$HC_n(G; k) \cong H_n(G; k) \oplus H_{n-2}(G; k) \oplus \cdots,$$

$$HP_n(G; k) \cong \prod_{p=-\infty}^{+\infty} H_{n+2p}(G; k),$$

$$HC_n^-(G; k) \cong \prod_{p=0}^{\infty} H_{n+2p}(G; k).$$

The Connes-Gysin sequences break up into short exact sequences, the B maps being zero. Finally, the maps $HC_n^- \to HH_n$ and $HP_n \to HC_n$ are the natural surjections given by the above formulas.

FIRST PROOF. The HH calculation is from 1.2.3, and the HC calculation was done by Karoubi on p. 382 of [**K**]. The HP calculation now follows from (1.4), and the HC^- calculation from 1.5.

SECOND PROOF. This is an extension of Karoubi's calculation in [**K**]. We adopt the notation of exercise 1.6 above; since $\text{Cyc}_{..}(kG; _{\varepsilon}kG)$ is a free G-resolution of $K.$, and since $\text{Cyc}_{..}(kG; _{\varepsilon}k) = G \setminus \text{Cyc}_{..}(kG; _{\varepsilon}kG)$, a hyperhomology calculation yields

$$HP_n(G; k) = \text{hyper } H_n(G; K.)$$

$$= \varprojlim \text{hyper } H_n(G; k \leftarrow 0 \leftarrow \cdots \leftarrow k \leftarrow \cdots)$$

$$= \prod H_{n+2p}(G; k).$$

In this calculation we used the easily checked fact that the system

$$\{\text{hyper } H_n(G; k \leftarrow \cdots)\}$$

satisfies the Mittag-Leffler condition, so that $\varprojlim^1 = 0$. The same formal calculation goes through for HC and HC^-.

REMARK 1.8. Although there is no *canonical* splitting of the projection from $HC_n^- \cong \prod H_{n+2p}(G; k)$ onto $HH_n \cong H_n(G; k)$, it is possible to select splittings which are natural in G. In fact, we can find splittings of the underlying chain map

(1.8.1) $\text{tot}_n = \text{tot}(\text{Cyc}_{\cdot\cdot}^-(kG; k))_n \to Cyc_n(kG; k).$

Here we have written k for the bimodule ${}_\varepsilon k$. This is essentially the content of Lemma II.3.2 in [G] and uses· acyclic models. First choose the canonical lift of $k = \text{Cyc}_0(kG; k)$. Next, let Z be infinite cyclic on z and choose a lifting of $1 \otimes z \in \text{Cyc}_1(kZ; k)$ to a cycle; this is possible by 1.7 because $HC_1^- \cong HH_1 \cong k$. Use this to find natural liftings of all $1 \otimes g$, hence a natural splitting $\text{Cyc}_1(kG; k) \to \text{tot}_1$. Similarly, $\text{Cyc}_p(kG; k)$ is a free k-module for $p \geq 2$, and is modeled by the case in which G is a free group on p generators. In this case we know by 1.7 that both complexes in 1.8.1 are exact at p, so there is no difficulty in inductively finding a natural map $\text{Cyc}_p(kG; k) \to \text{tot}_p$ compatible with the differentials.

2. Proof of Theorem A. In this section, we will take $k = \mathbb{Z}$, let A be a ring with unit, and fix $G = \text{GL}(A)$. For $n \geq 1$, let h denote the Hurewicz map:

$$K_n(A) = \pi_n B\text{GL}^+(A) \to H_n(B\text{GL}^+(A); \mathbb{Z}) \cong H_n(BG; \mathbb{Z}) \cong H_n(G; \mathbb{Z}).$$

Let $M_i(A)$ denote the ring of $i \times i$ matrices over A, $MA = \bigcup M_i(A)$, and let $\mathbb{Z}G \to MA$ denote the ring map induced from the inclusions of $\text{GL}_i(A)$ into $M_i(A)$. Finally, choose a splitting s of $HC_*^-(G; k) \to HH_*(G; k)$ using 1.7 and 1.8. There is a commutative diagram (for $n \geq 1$):

(2.1)

The split injections ι and the trace maps are defined in 1.3. The composite $K_n(A) \rightarrow HH_n(A)$ is the Dennis trace map as defined in [**D**], and the composite $K_n(A) \rightarrow HC_n^-(A)$ is Goodwillie's map γ of [**G**]. I claim that the composite $I\gamma : K_n(A) \rightarrow HP_n(A)$ is due to Karoubi.

PROPOSITION 2.2. *The composite of the map $K_n(A) \rightarrow HP_n(A)$ in (2.1) and the projection $HP_n(A) \rightarrow \varprojlim HC_{n+2p}(A)$ in (1.4) is the Chern class map defined by Karoubi in* [**K1**].

PROOF. Reinterpreting the construction in [**K1**] using

$$\varprojlim HC_{n+2p}(G; \mathbb{Z}) \cong HP_n(G; \mathbb{Z}) \cong \prod_{p=-\infty}^{+\infty} H_{n+2p}(G; \mathbb{Z}),$$

the Chern class of [**K1**] is the composite

$$K_n(A) \xrightarrow{h} H_n(G; \mathbb{Z}) \xrightarrow{\kappa} \prod_{p=-\infty}^{+\infty} H_{n+2p}(G; \mathbb{Z})$$

$$\cong HP_n(G; \mathbb{Z}) \rightarrow HP_n(A) \rightarrow \varprojlim HC_{n+2p}(A),$$

where the map $HP_n(G; \mathbb{Z}) \rightarrow HP_n(A)$ is from diagram (2.1). The key map here is κ, and it can be taken to be

$$H_n(G; \mathbb{Z}) \xrightarrow{s} \prod_{p=0}^{\infty} H_{n+2p}(G; \mathbb{Z}) \cong HC_n^-(G; \mathbb{Z}) \rightarrow HP_n(G; \mathbb{Z}).$$

With this choice, the Chern class map becomes the map in (2.1). $\square$

3. Calculations. The best way to understand the theory in the previous sections (and next) is to perform a few calculations. Throughout this section, we compute over $k = \mathbb{Z}$.

EXAMPLE 3.1. Let $R = \mathbb{Z}[t]$, so that $\Omega_R \cong R$ on dt and $\Omega_R/dR \cong \bigoplus(\mathbb{Z}/j)$ on generators $t^{j-1}dt$. From [**LQ**, 2.6] we see that

$$HC_n(\mathbb{Z}[t])] \begin{cases} \mathbb{Z}[t] & \text{if } n = 0, \\ \mathbb{Z} & \text{if } n \text{ is even, } n \geq 2, \\ \bigoplus(\mathbb{Z}/j) & \text{if } n \text{ is odd, } n \geq 1, \\ 0 & \text{if } n < 0. \end{cases}$$

Since $HH_n(\mathbb{Z}[t]) = 0$ for $n \geq 2$, it follows from 1.5 that

$$HP_n(\mathbb{Z}[t]) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even,} \\ \bigoplus(\mathbb{Z}/j) & \text{if } n \text{ is odd.} \end{cases}$$

In particular, note that $HP_*(\mathbb{Z}[t]) \neq HP_*(\mathbb{Z})$.

In contrast, we offer the more pleasing

LEMMA 3.2. *If $\mathbb{Q} \subset A$ then $HP_*(A[t]) \cong HP_*(A)$.*

PROOF. By [**Kass**, 4.3] with $B = \mathbb{Q}[t]$, we have

$$HC_n(A[t]) = HC_n(A) \oplus HH_n(A) \otimes t\mathbb{Q}[t]$$

as a comodule over $\mathbb{Z}[S]$. That is, $S = 0$ on the second term. The result follows from (1.4).

When I is an ideal we shall write $HX_*(A, I)$ for $HX_*(L)$, where the cyclic module L is the kernel of $\mathrm{Cyc}(A; A) \to \mathrm{Cyc}(A/I; A/I)$. Thus there are exact sequences

$$\cdots HX_{n+1}(A/I) \to HX_n(A, I) \to HX_n(A) \to HX_n(A/I) \cdots .$$

The easiest case of this is when $A = \mathbb{Q}[\varepsilon]$, $I = \mathbb{Q}\varepsilon$. From [**LQ**, (4.3)] we obtain

EXAMPLE 3.3.

$$HH_n(\mathbb{Q}[\varepsilon], \mathbb{Q}\varepsilon) = \begin{cases} \mathbb{Q} & \text{if } n \geq 0, \\ 0 & \text{if } n < 0, \end{cases}$$

$$HC_n(\mathbb{Q}[\varepsilon], \mathbb{Q}\varepsilon) = \begin{cases} \mathbb{Q} & \text{if } n \text{ is even}, n \geq 0, \\ 0 & \text{if not.} \end{cases}$$

Every map S is 0, and the Connes-Gysin sequence breaks up. From this and (1.4), we see that

$$HP_n(\mathbb{Q}[\varepsilon], \mathbb{Q}\varepsilon) = 0 \quad \text{all } n,$$

$$HC_n^-(\mathbb{Q}[\varepsilon], \mathbb{Q}\varepsilon) = \begin{cases} \mathbb{Q} & \text{if } n \text{ is odd}, n \geq 1, \\ 0 & \text{if not.} \end{cases}$$

Goodwillie proved in [**G**] that there are isomorphisms

$$K_n(\mathbb{Q}[\varepsilon], \mathbb{Q}\varepsilon) \xrightarrow[\gamma]{\cong} HC_n^-(\mathbb{Q}[\varepsilon], \mathbb{Q}\varepsilon) \xleftarrow[B]{\cong} HC_{n-1}(\mathbb{Q}[\varepsilon], \mathbb{Q}\varepsilon).$$

In fact, by [**Ogle**, §4] we know for every odd n that Loday's double bracket symbol $\langle\langle \varepsilon, \varepsilon, \ldots, \varepsilon \rangle\rangle$ in $K_n(\mathbb{Q}[\varepsilon], \mathbb{Q})$ is nonzero, because it maps nontrivially into $HH_n(\mathbb{Q}[\varepsilon])$. (This symbol is defined in [**L** and **Ogle**].)

EXAMPLE 3.4. When $A = \mathbb{Z}[\varepsilon]$ we encounter more delicate calculations. It is easy to directly compute that

$$HH_n(\mathbb{Z}[\varepsilon], \mathbb{Z}\varepsilon) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even}, n \geq 0, \\ \mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } n \text{ is odd}, n \geq 1, \\ 0 & \text{if } n < 0. \end{cases}$$

The element $\varepsilon \otimes \cdots \otimes \varepsilon$ of L_n is a cycle. If n is even it generates the $\mathbb{Z}$; if n is odd it generates the $\mathbb{Z}/2$. The $\mathbb{Z}$ for n odd is generated by $1 \otimes \varepsilon \otimes \cdots \otimes \varepsilon$. From [**LQ**, (4.3) or **Ogle**, 2.1] we get

$$HC_n(\mathbb{Z}[\varepsilon], \mathbb{Z}\varepsilon) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even}, n \geq 0, \\ \displaystyle\prod_{\substack{m=1 \\ m \text{ odd}}}^{n} (\mathbb{Z}/m \oplus \mathbb{Z}/2) & \text{if } n \text{ is odd}, n \geq 1, \\ 0 & \text{if } n < 0. \end{cases}$$

From [**Ogle**, 2.1], we see that the map $S : HC_n \to HC_{n-2}$ is zero if n is even, and the evident projection if n is odd. Thus

$$HP_n(\mathbb{Z}[\varepsilon], \mathbb{Z}\varepsilon) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ \displaystyle\prod_{\substack{m=1 \\ m \text{ odd}}}^{\infty} (\mathbb{Z}/m \oplus \mathbb{Z}/2) & \text{if } n \text{ is odd}. \end{cases}$$

From this, or by performing a similar calculation, we get

$$
HC_n^-(\mathbb{Z}[\varepsilon], \mathbb{Z}\varepsilon) = \begin{cases} 0 & \text{if } n \text{ is even,} \\[2mm] (\mathbb{Z} \oplus \mathbb{Z}/2) \oplus \displaystyle\prod_{\substack{m > n \\ m \text{ odd}}}^{\infty} (\mathbb{Z}/m \oplus \mathbb{Z}/2) & \text{if } n \text{ is odd, } n \geq 1, \\[4mm] \displaystyle\prod_{\substack{m=1 \\ m \text{ odd}}}^{\infty} (\mathbb{Z}/m \oplus \mathbb{Z}/2) & \text{if } n \text{ is odd, } n < 0. \end{cases}
$$

Now it is well known that $K_1(\mathbb{Z}[\varepsilon], \mathbb{Z}\varepsilon) = (1 + \mathbb{Z}\varepsilon)^* \cong \mathbb{Z}$ and that the Dennis trace map sends u to $u^{-1} \otimes u$. In particular, $u = 1 + \varepsilon$ maps to the cycle $(1 - \varepsilon) \otimes (1 + \varepsilon) \sim 1 \otimes \varepsilon - \varepsilon \otimes \varepsilon$ which under the given isomorphism $HH_1 \cong \mathbb{Z} \oplus \mathbb{Z}/2$ corresponds to $(1, 1)$. In $HC_1 \cong \mathbb{Z}/2$ the cycle $1 \otimes \varepsilon$ is zero but $\varepsilon \otimes \varepsilon$ is nonzero. We record the following consequence:

OBSERVATION 3.4.1. Goodwillie's map $\gamma \colon K_1(\mathbb{Z}[\varepsilon], \mathbb{Z}\varepsilon) \to HC_1^-(\mathbb{Z}[\varepsilon], \mathbb{Z}\varepsilon)$ is injective, because the Dennis trace map is. In addition, Karoubi's Chern class is nonzero, because
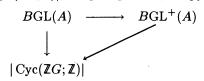
$$
\mathbb{Z} \cong K_1(\mathbb{Z}[\varepsilon], \mathbb{Z}\varepsilon) \xrightarrow{\text{ch}} HP_1(\mathbb{Z}[\varepsilon], \mathbb{Z}\varepsilon) \to HC_1(\mathbb{Z}[\varepsilon], \mathbb{Z}\varepsilon) \cong \mathbb{Z}/2
$$

is nonzero.

4. The space level. In this section, we shall realize the maps in (2.1) at the topological level. We shall start with the map h.

If X is a simplicial set, let $\mathbb{Z}X$ denote the simplicial abelian group which in degree n is the free abelian group on the set X_n. Almost by definition the homology of X, $H_*(X; \mathbb{Z})$, is the homotopy $\pi_*(\mathbb{Z}X) = HH_*(\mathbb{Z}X)$. There is a simplicial map $X \to \mathbb{Z}X$ (sending x to x), and it is classical [**May**] that the Hurewicz map is just $\pi_*(X) \to \pi_*(\mathbb{Z}X) = H_*(X; \mathbb{Z})$.

Now let G be a group, and let N be the nerve of G, i.e., the simplicial set whose realization $|N|$ is BG. Almost by definition, the simplicial abelian group $\mathbb{Z}N$ is $\mathrm{Cyc}(\mathbb{Z}G; \mathbb{Z})$. Specializing even further, let $G = \mathrm{GL}(A)$. Since addition makes the realization of any simplicial abelian group into an H-space, it follows that there is a map $B\mathrm{GL}^+(A) \to |\mathrm{Cyc}(\mathbb{Z}G; \mathbb{Z})|$, unique up to homotopy, such that

$$
\begin{array}{ccc} B\mathrm{GL}(A) & \longrightarrow & B\mathrm{GL}^+(A) \\ \downarrow & \swarrow & \\ |\mathrm{Cyc}(\mathbb{Z}G; \mathbb{Z})| & & \end{array}
$$

is homotopy commutative. From these remarks and the construction of h, we obtain

LEMMA 4.1. *The map* $B\mathrm{GL}^+(A) \to |\mathrm{Cyc}(\mathbb{Z}G; \mathbb{Z})|$ *induces the map* $h \colon K_n(A) \to H_n(\mathrm{GL}(A); \mathbb{Z})$.

Next, we turn to the maps in the lower part of (2.1). These are induced from maps of chain complexes of abelian groups. In order to obtain topological spaces, we use the Dold-Kan theorem, which states that the category of simplicial abelian groups is equivalent to the category **Ch** of chain complexes C. with $C_n = 0$ for $n < 0$. Under this correspondence, homotopy corresponds to homology. Unfortunately, the chain complexes involved are not in **Ch**, so we need to truncate.

If $C.$ is a chain complex, let $0 \setminus C.$ denote the subcomplex

$$(0 \setminus C)_n = \begin{cases} 0 & \text{if } n < 0, \\ \ker(C_0 \to C_{-1}) & \text{if } n = 0, \\ C_n & \text{if } n > 0. \end{cases}$$

Then $0 \setminus C.$ is in **Ch** and $H_n(0 \setminus C.) = H_n(C.)$ for $n \geq 0$. In fact, $0 \setminus$ is the reflection into **Ch**; if $P.$ is in **Ch** then

$$\mathrm{Hom}(P., C.) = \mathrm{Hom}(P., 0 \setminus C.).$$

If we write DK for the Dold-Kan equivalence from **Ch** to simplicial abelian groups, and $|L|$ for the realization of L, we have

$$\begin{bmatrix} \text{Chain} \\ \text{complexes} \end{bmatrix} \xrightarrow{0\setminus} \mathbf{Ch} \xrightarrow{DK} \begin{bmatrix} \text{Simplicial} \\ \text{abelian groups} \end{bmatrix} \xrightarrow{|\cdot|} \text{Spaces.}$$

To streamline notation, let us agree to write $X(C)$ for $|DK(0 \setminus C)|$. From the above remarks, we see that

$$H_n(C) = \pi_n X(C) \quad \text{for } n \geq 0.$$

In particular, $HC_n^-(A)$ and $HP_n(A)$ are the homotopy groups (for $n \geq 0$) of the spaces

$$X(\mathrm{tot}(\mathrm{Cyc}_{..}^-(A; A))) \quad \text{and} \quad X(\mathrm{tot}(\mathrm{Cyc}_{..}(A; A))).$$

Since all the maps in the bottom of (2.1) are induced in homology by chain maps, it follows that they may also be considered to be induced in homotopy by maps of spaces.

To get the map s of (2.1), recall from 1.8 that s is induced by a chain map (1.8.1). It follows that s is induced in homotopy by a map of spaces $X(\mathrm{Cyc}.(\mathbb{Z}G; \mathbb{Z})) \to X(\mathrm{tot}(\mathrm{Cyc}_{..}^-(\mathbb{Z}G; \mathbb{Z})))$, where $G = \mathrm{GL}(A)$. But there is a homotopy equivalence of simplicial abelian groups $\mathrm{Cyc}(\mathbb{Z}G; \mathbb{Z}) \to DK(\mathrm{Cyc}.(\mathbb{Z}G; \mathbb{Z}))$, so s is induced by the map of spaces

(4.2) $$|\mathrm{Cyc}(\mathbb{Z}G; \mathbb{Z})| \to X(\mathrm{tot}(\mathrm{Cyc}_{..}^-(\mathbb{Z}G; \mathbb{Z}))).$$

Putting 4.1 and (4.2) together, we see that we have proven the following proposition.

PROPOSITION 4.3. *The diagram* (2.1) *arises from applying* π_n *to the following diagram of spaces*:

$$BGL^+(A)$$
$$\downarrow$$
$$|\operatorname{Cyc}(\mathbb{Z}G;\mathbb{Z})|$$
$$\downarrow$$

$$
\begin{array}{ccccc}
X(\operatorname{Cyc}.(\mathbb{Z}G;\mathbb{Z})) & \longleftarrow & X(\operatorname{tot}\operatorname{Cyc}_{\cdot\cdot}^-(G;\mathbb{Z})) & \longrightarrow & X(\operatorname{tot}\operatorname{Cyc}_{\cdot\cdot}(G;\mathbb{Z})) \\
\Big\uparrow\Big\downarrow & & \Big\uparrow\Big\downarrow & & \Big\uparrow\Big\downarrow \\
X(\operatorname{Cyc}.(\mathbb{Z}G;\mathbb{Z}G) & \longleftarrow & X(\operatorname{tot}\operatorname{Cyc}_{\cdot\cdot}^-(\mathbb{Z}G;\mathbb{Z}G)) & \longrightarrow & X(\operatorname{tot}\operatorname{Cyc}_{\cdot\cdot}(\mathbb{Z}G;\mathbb{Z}G)) \\
\downarrow & & \downarrow & & \downarrow \\
X(\operatorname{Cyc}.(MA;MA)) & \longleftarrow & X(\operatorname{tot}\operatorname{Cyc}_{\cdot\cdot}^-(MA;MA)) & \longrightarrow & X(\operatorname{tot}\operatorname{Cyc}_{\cdot\cdot}(MA;MA)) \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
X(\operatorname{Cyc}.(A;A)) & \longleftarrow & X(\operatorname{tot}\operatorname{Cyc}_{\cdot\cdot}^-(A;A)) & \longrightarrow & X(\operatorname{tot}\operatorname{Cyc}_{\cdot\cdot}(A;A))
\end{array}
$$

We shall conclude this section with a description of the Connes-Gysin sequence at the space level. Let Σ denote suspension of chain complexes, so $H_n(\Sigma C) = H_{n-1}(C)$. Given a cyclic group L, consider the diagram of chain complexes (whose rows are *not* exact in the middle):

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{tot}(L_{\cdot\cdot}^-) & \longrightarrow & \operatorname{tot}(L_{\cdot\cdot}) & \longrightarrow & \Sigma^2\operatorname{tot}(L_{\cdot\cdot}^+) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L. & \longrightarrow & \operatorname{tot}(L_{\cdot\cdot}^+) & \longrightarrow & \Sigma^2\operatorname{tot}(L_{\cdot\cdot}^+) & \longrightarrow & 0
\end{array}
$$

As in [**LQ**, 1.6], these induce the Connes-Gysin sequences of 1.5 when we apply homology. Since X sends homology to homotopy, we have

PROPOSITION 4.4. *The following diagram of topological spaces commutes, and the rows are homotopy fibrations*:

$$
\begin{array}{ccccc}
X(\operatorname{tot}(L_{\cdot\cdot}^-)) & \longrightarrow & X(\operatorname{tot}(L_{\cdot\cdot})) & \longrightarrow & X(\Sigma^2\operatorname{tot}(L_{\cdot\cdot}^+)) \\
\downarrow & & \downarrow & & \| \\
X(L.) & \longrightarrow & X(\operatorname{tot}(L_{\cdot\cdot}^+)) & \longrightarrow & X(\Sigma^2\operatorname{tot}(L_{\cdot\cdot}^+))
\end{array}
$$

Moreover, the long exact sequences on homotopy are the Connes-Gysin sequences of 1.5.

REMARK 4.4.1. If C is in **Ch**, then $\Omega X(\Sigma C) \simeq X(C)$. Thus $X(C)$ is an infinite loop space. Agreeing that $\Omega^{-1}X(C)$ denotes the connected space $X(\Sigma C)$,

4.4 induces the map of homotopy fibrations

$$
\begin{array}{ccc}
\Omega^{-1}X(\mathrm{tot}(L_{\cdot\cdot}^{+})) & \longrightarrow & X(\mathrm{tot}(L_{\cdot\cdot}^{-})) & \longrightarrow & X(\mathrm{tot}(L_{\cdot\cdot})) \\
\| & & \downarrow & & \downarrow \\
\Omega^{-1}X(\mathrm{tot}(L_{\cdot\cdot}^{+})) & \longrightarrow & X(L_{\cdot}) & \longrightarrow & X(\mathrm{tot}(L_{\cdot\cdot}^{+}))
\end{array}
$$

5. The maps ch **and** ν**.** In this section, we shall factor Karoubi's Chern class map for $\mathbb{Q}$-algebras through KV-theory. Breaking K-theory up into the two pieces KV and nil K, we will also induce a map (for $\mathbb{Q} \subset A$) ν: nil $K_n(A) \to HC_{n-1}(A)$ compatible with ch, proving Theorem C.

Example 3.4 above shows that the assumption $\mathbb{Q} \subset A$ is necessary; $KV_1(\mathbb{Z}[\varepsilon], \mathbb{Z}\varepsilon)$ $= 0$ holds, yet as observed in (3.4.1) the map $K_1 \to HP_1$ is nonzero. In fact, the difficulty may be traced to the fact that HP_* is not homotopy invariant unless we restrict to $\mathbb{Q}$-algebras. This problem is illustrated by Example 3.1 and Lemma 3.2.

We need to recall the construction of KV from [**And**]. Given a ring A, let ΔA denote the simplicial ring with

$$(5.1) \qquad \Delta_n A = A[t_0, \ldots, t_n]/(\Sigma t_i = 1) \cong A[t_1, \ldots, t_n]$$
$$d_i(t_i) = 0 \quad \text{and} \quad s_i(t_i) = t_i + t_{i+1}.$$

If $i \neq j$ we have $d_i(t_j) = t_j$ or t_{j-1} and $s_i(t_j) = t_j$ or t_{j+1}. Heuristically, $\Delta_n A$ is the ring of functions on the n-simplex, and the face and degeneracy maps are determined by geometry.

DEFINITIONS 5.2. Set $K(A) = BGL^+(A)$ and $KV(A) = |BGL^+(\Delta A)|$ (=the geometric realization of the simplicial space $K(\Delta A)$). Note that $K_n(A) = \pi_n K(A)$ and $KV_n(A) = \pi_n KV(A)$ for $n \geq 1$ [**And**]. Let nil $K(A)$ denote the homotopy fiber of $K(A) \to KV(A)$. For $n \geq 1$, we define

$$\mathrm{nil}\, K_n(A) = \pi_n \, \mathrm{nil}\, K(A).$$

The long exact homotopy sequence for the fibration nil $K \to K \to KV$ is thus:

$$\cdots \to \mathrm{nil}\, K_2(A) \to K_2(A) \to KV_2(A) \to \mathrm{nil}\, K_1(A) \to K_1(A) \to KV_1(A) \to 0$$

REMARK 5.2.1. One usually defines $KV_n(A) = K_n(A)$ for $n \leq 0$. To be consistent, we would therefore set nil $K_n(A) = 0$ for $n \leq 0$. Such a convention would allow us to extend Theorem C to all values of n. However, such an extension introduces extra technical complications, as in [**W**], so we shall forego discussing such an extension here.

REMARK 5.2.2. The above definition of nil $K_n(A)$ is different from that of [**W1**, 4.1], where I defined nil $K_n(A)$ to be the kernel of $K_n(A) \to KV_n(A)$. I will elaborate on this point in §6.

Now recall from §4 that there is a topological space $Y(A) = X(\mathrm{tot}\, \mathrm{Cyc}_{\cdot\cdot}(A; A))$, depending functorially on A, with the property that $\pi_n Y(A) = HP_n(A)$. By 4.3, we have the diagram

$$
\begin{array}{ccc}
K(A) = BGL^+(A) & \xrightarrow{I\gamma} & Y(A) \\
\downarrow \qquad\qquad \downarrow & & \downarrow \\
KV(A) = |BGL^+(\Delta A)| & \xrightarrow{|I\gamma\Delta|} & |Y(\Delta A)|
\end{array}
$$

It can be seen from Examples 3.1 and 3.4 above that $Y(A) \to |Y(\Delta A)|$ is not a homotopy equivalence for $A = \mathbb{Z}[\varepsilon]$. However, we do have

PROPOSITION 5.3. *If* $\mathbb{Q} \subset A$ *then* $Y(A) \to |Y(\Delta A)|$ *is a homotopy equivalence. Hence the map* $I\gamma : K(A) \to Y(A)$ *factors through a map* ch: $KV(A) \to Y(A)$.

PROOF. By 3.2 above, $Y(A) \to Y(\Delta_n A)$ is a homotopy equivalence for each n. It is well known that this implies the result. (See [**W**, 1.7] for some references.)

COROLLARY 5.4. *If* $\mathbb{Q} \subset A$ *then Karoubi's Chern class map factors through a map* ch: $KV_n(A) \to HP_n(A)$.

COROLLARY 5.5. *If* $\mathbb{Q} \subset A$ *there is a map of homotopy fibrations*

$$
\begin{array}{ccccc}
\mathrm{nil}\,K(A) & \longrightarrow & K(A) & \longrightarrow & KV(A) \\
\downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{ch}} \\
\Omega^{-1}X(\mathrm{tot}\,\mathrm{Cyc}^{+}_{\cdot\cdot}(A;A)) & \longrightarrow & X(\mathrm{tot}\,\mathrm{Cyc}^{-}_{\cdot\cdot}(A;A)) & \longrightarrow & X(\mathrm{tot}\,\mathrm{Cyc}_{\cdot\cdot}(A;A)) \\
\| & & \downarrow & & \downarrow \\
\Omega^{-1}X(\mathrm{tot}\,\mathrm{Cyc}^{+}_{\cdot\cdot}(A;A)) & \longrightarrow & X(\mathrm{Cyc}_{\cdot}(A;A)) & \longrightarrow & X(\mathrm{tot}\,\mathrm{Cyc}^{+}_{\cdot\cdot}(A;A))
\end{array}
$$

PROOF. The map ν is induced from γ and ch. The bottom two rows are homotopy fibrations by 4.4.1.

DEFINITION 5.6. For $n \geq 1$, let ν denote π_n applied to the map ν of 5.5. Since $\pi_n X(\mathrm{tot}\,\mathrm{Cyc}^{+}_{\cdot\cdot}(A;A)) = HC_n(A)$, we have defined a map

$$\nu : \mathrm{nil}\,K_n(A) \to HC_{n-1}(A).$$

REMARK 5.7. If A is not a $\mathbb{Q}$-algebra, we still have the map

$$\mathrm{nil}\,K_n(A) \to \mathrm{nil}\,K_n(A \otimes \mathbb{Q}) \xrightarrow{\nu} HC_{n-1}(A \otimes \mathbb{Q}) \cong HC_{n-1}(A) \otimes \mathbb{Q}.$$

PROOF OF THEOREM C. Diagram 5.5 gives rise to maps of long exact sequences on homotopy. This diagram is the diagram of Theorem C.   Q.E.D.

6. Nil K-theory. In §5 we defined groups $\mathrm{nil}\,K_n(A)$ fitting into the long exact sequence

(6.1) $\qquad \cdots K_{n+1}(A) \to KV_{n+1}(A) \to \mathrm{nil}\,K_n(A) \to K_n(A) \to KV_n(A) \cdots .$

Note that $\mathrm{nil}\,K_n(A) = 0$ if A is regular, for then $K_n(A) = KV_n(A)$. In this section, we establish some of $\mathrm{nil}\,K$'s properties, including

THEOREM B. *If* $\mathbb{Q} \subset A$ *then* $\mathrm{nil}\,K_n(A)$ *is a* $\mathbb{Q}$-*vector space for all* n. *If* $p^n = 0$ *in* A *then* $\mathrm{nil}\,K_n(A)$ *is a* p-*group for all* n.

EXAMPLE 6.2 [**W1**, 4.3–4.4]. Let $k \to K$ be a map of regular rings, and set $B = k \oplus tK[t]$, $C = k \oplus tK[[t]]$. Then $KV_1(B) \cong K_1(k)$,

$$\mathrm{nil}\,K_1(B) \cong \mathrm{nil}\,K_1(C) \cong \Omega_{K/k}$$

and the following sequences are exact:

$$0 \to \mathrm{nil}\,K_1(B) \to K_1(B) \to KV_1(B) \to 0,$$
$$K_2(C) \to KV_2(C) \to \mathrm{nil}\,K_1(C) \to 0.$$

Note that the definition of nil K_* in [**W1**, 4.1] is a quotient of the nil K_* defined in this paper, so that 6.2 is not in disagreement with [**W1**, 4.4]. Similarly, the remark in [**W1**] that nil $K_1(A)$ is the image of $NK_1(A) \to K_1(A)$ becomes incorrect if we use our new definition; there is an exact sequence

(6.3) $$N^2 K_1(A) \to NK_1(A) \to \mathrm{nil}\, K_1(A) \to 0.$$

This is a special case of the following more general result.

THEOREM 6.4. *There is a first quadrant spectral sequence, defined for $p \geq 1, q \geq 1$,*

$$E^1_{pq} = N^p K_q(A) \Longrightarrow \mathrm{nil}\, K_{p+q-1}(A).$$

$$\cdots$$

$$
\begin{array}{llll}
0 & NK_3 \leftarrow N^2 K_3 \\
0 & NK_2 \leftarrow N^2 K_2 \leftarrow \cdots \\
0 & NK_1 \leftarrow N^2 K_1 \leftarrow N^3 K_1 \cdots \\
0 & 0 & 0 & 0
\end{array}
$$

Here $N^p K_q(A)$ is the intersection of the kernels of the maps

$$(t_i = 0) : K_q(A[t_1, \ldots, t_p]) \to K_q(A[t_1, \ldots, \hat{t_i}, \ldots, t_p]), \qquad i = 1, \ldots, p,$$

and the d^1 map is induced from the ring map

$$(t_p = 1 - \textstyle\sum t_i) : A[t_1, \ldots, t_p] \to A[t_1, \ldots, t_{p-1}].$$

PROOF OF THEOREM B. This is immediate from Theorem 6.4 and the fact (q.v. [**W4**, 3.3]) that the groups $N^i K_q(A)$ are $\mathbb{Q}$-modules (resp., p-groups) when A is an algebra over $\mathbb{Q}$ (resp., some $\mathbb{Z}/p^n$) and $i \geq 1$.

To establish 6.4 above, we shall construct nil $K(A)$ at the level of simplicial CW spectra, following [**W3**, §3]. The construction of [**W3**, 2.2] yields a spectrum $\mathbf{K}(A)$ with $\Omega^\infty \mathbf{K}(A) \cong BGL^+(A)$. The simplicial spectrum $\mathbf{K}(\Delta A)$ has total spectrum $|\mathbf{K}(\Delta A)|$ whose homotopy groups yield $KV_*(A)$ by [**W4**, 2.1]. Now consider the cofibers $\mathbf{K}(\Delta_n A)/\mathbf{K}(A)$; they assemble to form a simplicial spectrum $\mathbf{K}(\Delta A)/\mathbf{K}(A)$. Since cofibrations and fibrations of spectra are the same thing,

$$\mathbf{K}(A) \to \mathbf{K}(\Delta_n A) \to \mathbf{K}(\Delta_n A)/\mathbf{K}(A)$$

is a homotopy fibration of connected spectra for each n. Hence we deduce

LEMMA 6.5. *The sequence of total spectra*

$$\mathbf{K}(A) \to |\mathbf{K}(\Delta A)| \to |\mathbf{K}(\Delta A)/\mathbf{K}(A)|$$

is a homotopy fibration for each n.

DEFINITION 6.6. The spectrum $\Sigma^{-1}|\mathbf{K}(\Delta A)/\mathbf{K}(A)|$ will be denoted nil $\mathbf{K}(A)$ and $\pi_n(\mathrm{nil}\, \mathbf{K}(A))$ will be denoted nil $K_n(A)$. By 6.5, this agrees with the definition of nil $K_n(A)$ given in 5.2 above, and nil $K(A)$ is the space Ω^∞ nil $\mathbf{K}(A)$.

PROOF OF THEOREM 6.4. This follows immediately from [**W3**, 3.2] applied to the simplicial spectrum $\mathbf{K}(\Delta A)/\mathbf{K}(A)$ with $D = \Sigma^\infty S^0$.

It is clear from [**W3**, §3] that variants of Theorem 6.4 and Theorem B hold in relative and doubly relative contexts as well. The following examples illustrate this phenomenon.

EXAMPLE 6.7. When I is a nilpotent ideal in A, we know from [**W2**, 2.2] that $KV_n(A, I) = 0$. Hence nil $K_n(A, I) \cong K_n(A, I)$ for all n. Theorem B in this case follows from [**W3**, 1.4] and [**W4**, 5.4].

EXAMPLE 6.8. When $A = A_0 \oplus A_1 \oplus \cdots$ is a graded ring, and $I = I_1 \oplus \cdots$ a graded ideal, we know that $KV_n(A, I) = 0$ because $KV_n(A) = KV_n(A_0)$. Hence

$$\operatorname{nil} K_n(A, I) \cong K_n(A, I) \quad \text{for all } n.$$

Theorem B in this case follows from [**W4**, 3.7].

EXAMPLE 6.9 (EXCISION). Let I be an ideal of A mapped by $A \to B$ isomorphically onto an ideal of B. Then we can consider the doubly relative K-groups. By [**W**, 2.6], $KV_n(A, B, I) = 0$, so

$$\operatorname{nil} K_n(A, B, I) \cong K_n(A, B, I) \quad \text{for all } n.$$

Theorem B in this case follows from [**W3**, 1.5].

EXAMPLE 6.10. Let A be a 1-dimensional noetherian seminormal ring, such as the coordinate ring of a curve with normal crossings. The normalization B of A is Dedekind, and the conductor ideal I is radical in B. Hence nil $K_*(B)$ = nil $K_*(A/I)$ = nil $K_*(B/I)$ = 0. It follows that nil $K_*(B, I) = 0$ and that nil $K_*(A, I) = $ nil $K_*(A)$, e.g., from the long exact sequence

$$\cdots \operatorname{nil} K_{n+1}(A/I) \to \operatorname{nil} K_n(A, I) \to \operatorname{nil} K_n(A) \to \operatorname{nil} K_n(A/I) \cdots.$$

From the long exact sequence

$$\cdots \operatorname{nil} K_{n+1}(B, I) \to \operatorname{nil} K_n(A, B, I) \to \operatorname{nil} K_n(A, I) \to \operatorname{nil} K_n(B, I) \cdots$$

and 6.9, we see that in this case

$$\operatorname{nil} K_n(A) \cong K_n(A, B, I).$$

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